Tukey reduction

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Outline of Topics

All the unattributed results are due to Todorcevic and myself.

Tukey reduction and basic orders

Tukey reduction

A **directed order** $(D, \leq)$ is a partial order such that for each $x, y \in D$ there is $z \in D$ with $x, y \leq z$.

A set $A \subseteq D$ is called **bounded** if there is $x \in D$ such that $y \leq x$ for each $y \in A$.

A set $A \subseteq D$ is **cofinal** if for each $x \in D$ there $y \in A$ with $x \leq y$.

D and E directed orders.

A function $f\colon D \to E$ is called **Tukey** if preimages under f of sets bounded in E are bounded in D.
We write

$$D \leq_T E$$

if there is a Tukey function from D to E.

If D and E are Tukey reducible to each other, we say that they are **Tukey equivalent**, and we write

$$D \equiv_T E.$$

Theorem (Tukey)

Let D and E be directed order. Then $D \equiv_T E$ if and only if D and E can be embedded as cofinal subsets of a directed order.

Dual point of view: A function $g\colon E \to D$ is **convergent** if images under g of sets cofinal in E are cofinal in D.

For two directed orders D and E, there is a Tukey function from D to E if and only if there is a convergent function from E to D.

Examples.

$$\mathbb{N} <_T \mathbb{N}^{\mathbb{N}}$$

$$\mathbb{N} \not\leq_T \omega_1,\ \omega_1 \not\leq_T \mathbb{N}$$

Connection with cardinal invariants.

$$\mathrm{add}(D) = \text{ minimal cardinality of an unbounded subset of } D$$

$$\mathrm{cof}(D) = \text{ minimal cardinality of a cofinal subset of } D.$$

$$D \leq_T E \Longrightarrow \mathrm{add}(E) \leq \mathrm{add}(D) \text{ and } \mathrm{cof}(D) \leq \mathrm{cof}(E).$$

Basic orders

A directed order D is called **basic** if

- D is a separable metric space;
- each two elements of D have the least upper bound and the operation of taking the least upper bound is a continuous function from $D \times D$ to D;
- each bounded sequence has a convergent subsequence;
- each convergent sequence has a bounded subsequence.

Examples of basic orders.

1. $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$

2. NWD all *closed nowhere dense* subsets of $2^{\mathbb{N}}$ taken with inclusion as the directed order relation
View NWD as a subset of the compact space $\mathcal{K}(2^{\mathbb{N}})$ with the Vietoris topology.

3. ℓ_1 all subsets x of $\mathbb{N}$ with

$$\sum_{n \in x} \frac{1}{n+1} < \infty$$

taken with inclusion as the directed order relation
View ℓ_1 with the topology given by the following metric

$$d(x, y) = \sum_{n \in x \triangle y} \frac{1}{n+1}.$$

A separable metric is called **analytic** if it is a continuous image of a Polish space. For example, all Borel subsets of Polish spaces are analytic.

Basic orders whose underlying topology is analytic are called **analytic basic orders**.

All the examples above are analytic basic orders.

Analytic basic orders form an initial class of basic orders.

Theorem

Let D and E be basic orders. If E is analytic and $D \leq_T E$, then D is analytic.

Theorem

Let D be a basic order. If the topology on D is analytic, then it is Polish.

Theorem

Let D and E be analytic basic orders. If $D \leq_T E$, then there exist a Tukey function from D to E that is measurable with respect to the σ-algebra generated by analytic sets.

The interesting analytic basic orders are the non-locally compact ones: $\mathbb{N}^{\mathbb{N}}$, NWD, ℓ_1; not $\mathbb{N}$.

Proposition

Let D be an analytic non-locally compact basic order. Then $\mathbb{N}^{\mathbb{N}} \leq_T D$.

Back to cardinal invariants.

MGR = all meager subsets of $2^{\mathbb{N}}$ taken with inclusion

NULL = all Lebesgue measure zero subsets of $[0,1]$ taken with inclusion

These are directed orders that are not basic orders. Cardinal invariants

$$\mathrm{add/cof}(\mathrm{MGR}) \text{ and } \mathrm{add/cof}(\mathrm{NULL})$$

are of interest.

A set is σ-**bounded** if it is a countable union of bounded sets.

D, E directed orders

$$D \leq_T^{\omega} E$$

if there is a function $D \to E$ such that preimages of σ-bounded sets are σ-bounded.

$$D \equiv_T^{\omega} E$$

if both $D \leq_T^{\omega} E$ and $E \leq_T^{\omega} D$.

Note: $D \leq_T E$ implies $D \leq_T^{\omega} E$.

$\mathrm{add}^{\omega}(D) =$ minimal cardinality of a non-σ-bounded subset of D.

$$D \leq_T^{\omega} E \Longrightarrow \mathrm{add}^{\omega}(E) \leq \mathrm{add}^{\omega}(D),\ \mathrm{cof}(D) \leq \max(\omega, \mathrm{cof}(E)).$$

Theorem (Bartoszyński, Raissonier–Stern, Fremlin)

$\mathrm{MGR} \equiv_T^\omega \mathrm{NWD}$ *and* $\mathrm{NULL} \equiv_T^\omega \ell_1$.

So

$$\mathrm{add}(\mathrm{MGR}) = \mathrm{add}^\omega(\mathrm{NWD}),\ \mathrm{cof}(\mathrm{MGR}) = \mathrm{cof}(\mathrm{NWD})$$

$$\mathrm{add}(\mathrm{NULL}) = \mathrm{add}^\omega(\ell_1),\ \mathrm{cof}(\mathrm{NULL}) = \mathrm{cof}(\ell_1).$$

So $\mathrm{NWD} \leq_T \ell_1$ *would* give

$$\mathrm{add}(\mathrm{NULL}) \leq \mathrm{add}(\mathrm{MGR}) \ \text{ and } \ \mathrm{cof}(\mathrm{MGR}) \leq \mathrm{cof}(\mathrm{NULL}).$$

Ideals

The main class of examples of basic orders are ideals taken with inclusion.

The world is divided into a **compact part** (σ-ideals, category leaf) and a **discrete part** (P-ideals, measure leaf).

σ-ideals

X a compact metric space

$\mathcal{K}(X)$ = all compact subsets of X with the Vietoris topology

$\mathcal{K}(X)$ is a compact metric space.

A set $\mathcal{I} \subseteq \mathcal{K}(X)$ is a σ-**ideal of compact sets** if it is closed under taking compact subsets and countable compact unions.

A σ-ideal of compact sets with inclusion and the topology inherited from $\mathcal{K}(X)$ is a basic order.

Kechris–Louveau–Woodin: a σ-ideal $\mathcal{I}$ of compact sets is locally compact if and only if $\mathcal{I} = \mathcal{K}(U)$ for some open set $U \subseteq X$.

Convention: a σ-**ideal** is an analytic, non-locally compact σ-ideal of compact subsets of a compact metric space.

A σ-ideal $\mathcal{I}$ has **property** $(*)$ if for each sequence (K_n) of sets in $\mathcal{I}$ there is a G_δ subset G of X such that $\bigcup_n K_n \subseteq G$ and all compact subsets of G are in $\mathcal{I}$.

Fact of nature: all naturally occurring σ-ideals have $(*)$.

Examples.

1. $\mathbb{N}^{\mathbb{N}}$ is Tukey equivalent to the σ-ideal with $(*)$ $\mathcal{K}([0,1] \setminus \mathbb{Q})$.

2. NWD is a σ-ideal with $(*)$.

3. Mátrai: there is a σ-ideal without $(*)$.

I found the following example $\mathcal{I}_0$.

Consider $\bar{s} = (s_0, s_1, \dots)$ infinite or finite with an even number of entries, each s_i is a function from a non-empty finite subset of $\mathbb{N}$ to 2, for each i, $\mathrm{dom}(s_i) < \mathrm{dom}(s_{i+1})$.

Let $\mathcal{R}$ be the set of all such sequences.

For $\bar{s} \in \mathcal{R}$, define

$$[\bar{s}] = \{x \in 2^{\mathbb{N}} \colon s_{2i} \subseteq x \text{ or } s_{2i+1} \subseteq x \text{ for each } i\}.$$

Define

$$\mathcal{I}_0 = \{K \in \mathcal{K}(2^{\mathbb{N}}) \colon K \cap [\bar{s}] \text{ is nowhere dense in } [\bar{s}] \text{ for each } \bar{s} \in \mathcal{R}\,\}.$$

$\mathcal{I}_0$ is a σ-ideal without $(*)$.

P-ideals

A set $I \subseteq \mathcal{P}(\mathbb{N})$ is a **P-ideal of subsets of** $\mathbb{N}$ if it is closed under taking finite unions and subsets and for each sequence $x_n \in I$, $n \in \mathbb{N}$, there is $x \in I$ such that $x_n \setminus x$ is finite for each n.

View I as a subspace of the compact metric space $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$

S.: I is an analytic P-ideal of subsets of $\mathbb{N}$ if and only if there exists a lower semicontinuous submeasure $\phi\colon \mathcal{P}(\mathbb{N}) \to [0,\infty]$ such that

$$I = \mathrm{Exh}(\phi) = \{x \in \mathcal{P}(\mathbb{N})\colon \lim_n \phi(x \setminus n) = 0\}.$$

Such an I becomes a Polish space with the **submeasure topology** given by the metric

$$d_\phi(x,y) = \phi(x \triangle y).$$

An analytic P-ideal of subsets of $\mathbb{N}$ taken with inclusion and with its submeasure topology is an analytic basic order.

S.–Todorcevic: if an ideal $I \subseteq \mathcal{P}(\mathbb{N})$ taken with inclusion and with a topology τ containing the topology inherited from $\mathcal{P}(\mathbb{N})$ is an analytic basic order, then I is an analytic P-ideal and τ is the submeasure topology.

S.: an analytic P-ideal I of subsets of $\mathbb{N}$ is locally compact with its submeasure topology if and only if $I = \{x \in \mathcal{P}(\mathbb{N})\colon x \cap a \text{ is finite}\}$ for some $a \subseteq \mathbb{N}$.

Convention: a **P-ideal** is an analytic, non-locally compact P-ideal of subsets of $\mathbb{N}$.

$I = \mathrm{Exh}(\phi)$ a P-ideal for a lower semicontinuous submeasure ϕ.

I **density-like** if for each $\epsilon > 0$ there is $\delta > 0$ such that for each sequence (x_n) of sets in I with $\phi(x_n) < \delta$ there are $n_0 < n_1 < n_2 < \cdots$ with

$$\phi(\bigcup_k x_{n_k}) < \epsilon.$$

Examples.

1. $\mathbb{N}^{\mathbb{N}}$ is Tukey equivalent to the density-like P-ideal

$$\emptyset \times \mathrm{Fin} = \{x \in \mathcal{P}(\mathbb{N} \times \mathbb{N})\colon \forall m\ \{n\colon (m,n) \in x\} \text{ is finite}\}.$$

2. The ideal

$$\mathcal{Z}_0 = \{x \in \mathcal{P}(\mathbb{N})\colon \lim_n \frac{|x \cap (n+1)|}{n+1} = 0\}$$

is a density-like P-ideal.
A lower semicontinuous submeasure for $\mathcal{Z}_0$:

$$\phi_0(x) = \sup_n \frac{|x \cap (n+1)|}{n+1}.$$

3. ℓ_1 is a P-ideal that is not density-like.
A lower semicontinuous submeasure for ℓ_1:

$$\phi_1(x) = \sum_{n \in x} \frac{1}{n+1}.$$

Structure of Tukey reduction among ideals

Within classes

Theorem (Louveau–Veličković, Todorcevic)

ℓ_1 *is Tukey largest among P-ideals.*

Is there a Tukey largest σ-ideal?

Theorem (Louveau–Veličković)

There is an embedding of the partial order $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ with almost inclusion into the class of P-ideals with with Tukey reduction.

Is the analogous result true for σ-ideals?

Theorem (S.)

NWD *is Tukey largest among* σ*-ideals with* $(*)$.

Is there a Tukey largest density-like P-ideal?

Across classes

There are essentially no Tukey reduction from the P-ideals to σ-ideals.

Theorem

If I is a P-ideal, $\mathcal{I}$ a σ-ideal, and $I \leq_T \mathcal{I}$, then I is isomorphic to $\emptyset \times \mathrm{Fin}$, so $I \equiv_T \mathbb{N}^{\mathbb{N}}$.

So if a P-ideal is Tukey equivalent to a σ-ideal, then they are both Tukey equivalent to the smallest analytic non-locally compact basic order $\mathbb{N}^{\mathbb{N}}$.

Among examples

Among the concrete examples defined above,

$$\mathbb{N}^{\mathbb{N}},\ \mathrm{NWD},\ \mathcal{I}_0,\ \mathcal{Z}_0,\ \ell_1,$$

the structure of Tukey reduction is completely known.

Theorem

(i) (Isbell, Fremlin, Louveau–Veličković)

$$\mathbb{N}^{\mathbb{N}} <_T \mathcal{Z}_0 <_T \ell_1$$

(ii) (Fremlin, Moore–Solecki)

$$\mathbb{N}^{\mathbb{N}} <_T \mathrm{NWD} <_T \mathcal{I}_0$$

Theorem

(i) (Bartoszyński, Raissonier–Stern, Fremlin) NWD $<_T \ell_1$

(ii) (Mátrai, Solecki–Todorcevic) NWD $\not\leq_T \mathcal{Z}_0$

(iii) (Mátrai) $\mathcal{I}_0 \not\leq_T \ell_1$

From (i) we get

$$\mathrm{add}(\mathrm{NULL}) \leq \mathrm{add}(\mathrm{MGR}) \text{ and } \mathrm{cof}(\mathrm{MGR}) \leq \mathrm{cof}(\mathrm{NULL}).$$

Shadow of NWD

Recall: $\mathrm{NWD} \leq_T \ell_1$ and $\mathrm{NWD} \not\leq_T \mathcal{Z}_0$.

Theorem

Let I be a density-like P-ideal. Then $\mathrm{NWD} \not\leq_T I$.

Characterize those P-ideals I for which $\mathrm{NWD} \leq_T I$.

Extracting an ordinal out of a P-ideal

I a P-ideal

$I = \mathrm{Exh}(\phi)$, for a lower semicontinuous submeasure ϕ

Given a sequence (x_n) of sets in I and $\epsilon > 0$, the set

$$\{b \subseteq \mathbb{N} \colon \phi(\bigcup_{n\in b} x_n) \leq \epsilon\} \subseteq 2^{\mathbb{N}}$$

is compact.

Let **height** of this set be ω_1 if it contains an infinite set.

If it consists of finite sets only, let its **height** be α, where α is such that its Cantor–Bendixson rank is $\alpha + 1$.

Let ϵ, $\delta > 0$ and $\alpha \in \omega_1$ be given.

$P_{\epsilon,\delta}(\alpha)$ **holds** if for every sequence (x_n) of sets in I with $\phi(x_n) < \delta$

$$\mathrm{height}(\{b\colon \phi(\bigcup_{n\in b} x_n) \leq \epsilon\}) \geq \alpha.$$

Let

$$\mathrm{ht}(I) = \mathsf{min}\{\alpha \in \omega_1 \colon \exists \epsilon > 0\, \forall \delta > 0\ P_{\epsilon,\delta}(\alpha) \text{ fails}\}$$

if the set on the right hand side is non-empty, and let

$$\mathrm{ht}(I) = \omega_1$$

otherwise.

Proposition

Let I be a P-ideal. Then

(i) $\mathrm{ht}(I)$ *does not depend on the choice of submeasure* ϕ *with* $I = \mathrm{Exh}(\phi)$;

(ii)
$$\mathrm{ht}(I) = \omega_1 \text{ or } \mathrm{ht}(I) = \omega^{\omega^\alpha} \text{ for some } \alpha < \omega_1.$$

A characterization of P-ideals with the largest and smallest values of height

Theorem

Let I be a P-ideal. Then

(i) $\mathrm{ht}(I) = \omega_1$ *if and only if I is density-like;*

(ii) $\mathrm{ht}(I) = \omega$ *if and only if* $I \equiv_T \ell_1$.

Height is an invariant of Tukey reduction.

Theorem

Let I, J be P-ideals. If $I \leq_T J$, then $\mathrm{ht}(J) \leq \mathrm{ht}(I)$.

Is there an ordinal α such that

$$\mathrm{NWD} \leq_T I \text{ if and only if } \mathrm{ht}(I) \leq \alpha\,?$$